The Divine Spark Complexity Law: Divine Logos, Mathematical Actualization, Relation Rank, and the Polynomial-to-Exponential Threshold in Finite Semilattices

Douglas C. Youvan

doug@youvan.com www.youvan.ai

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Modern mathematics and physics routinely presuppose identity, distinction, relation, law, truth, and intelligibility without providing an ontological ground for them. This paper argues that these conditions are grounded in the Divine Logos and that mathematical discovery is finite participation in a prior intelligible order. Within this Logos-centered framework, defining relations actualize determinate mathematical object-worlds from lawful spaces of possibility. A finite algebraic theorem makes this principle exact. For a finite meet-semilattice A, let $\gamma(A)$ be its minimum generator rank, $\sigma_2(A)$ its minimum relation rank, and $\nu_2(A)$ its minimum two-element subdirect dimension. Define $M_2(g,s)$ as the maximum $\nu_2(A)$ among semilattices with $\gamma(A) = g$ and $\sigma_2(A) \leq s$. Every indispensable binary coordinate has support size at most $s + 1$, because one irreducible root requires at most one permanent repair per defining relation. Consequently, $\nu_2(A) \leq \Sigma_{j=1}^{s+1} C(g,j)$, while balanced constructions give $M_2(g,s) = \Theta(g^{s+1})$. For one relation, $M_2(g,1) = \lfloor (g^2 + g + 1)/3 \rfloor$. Sublinear relation rank forces subexponential dimension; exponential dimension requires linear relation density.

Keywords: Divine Logos, Divine spark, origin of mathematics, philosophy of mathematics, mathematical intelligibility, mathematical actualization, mathematical objecthood, finite meet-semilattices, generator rank, relation rank, two-element subdirect dimension, fixed-core representation, Spark-Degree Law, root-and-repair lemma, extremal algebra, polynomial-to-exponential threshold, Logos-grounded mathematical physics.

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https://orcid.org/0009-0004-4206-4253

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1. The Foundational Crisis in Mathematics and Physics

1.1 Technical Power Without Ontological Ground

Modern mathematics possesses extraordinary formal power but no agreed account of its own intelligibility. It can define structures, prove consequences, and generate models, yet it ordinarily treats identity, distinction, relation, consistency, and truth as unexplained primitives.

Physics inherits the same problem. It discovers mathematical regularities but does not explain why physical reality is lawful, why those laws are mathematically expressible, or why human reason can recognize them.

The deficiency is not computational. It is ontological.

1.2 Why Formal Systems Cannot Ground Themselves

A formal system presupposes:

1. distinguishable symbols;
2. stable identity through transformation;
3. admissible rules;
4. valid inference;
5. a distinction between truth and error.

These conditions cannot be generated by the formal system without circularity. Any attempted derivation already employs the identities, relations, and logical norms it seeks to explain.

Thus no equation explains the possibility of equations. No algorithm explains the prior intelligibility of algorithms. Formal mathematics begins within an order it does not itself create.

1.3 Fragmented Secular Foundations

The major secular philosophies preserve partial truths.

Platonism preserves mathematical objectivity but leaves the mathematical realm ungrounded.

Formalism preserves rigor but reduces truth to rule-governed derivability.

Structuralism correctly emphasizes relation but does not explain why lawful structure exists.

Constructivism clarifies admissible formation but presupposes an intelligible constructor and valid operations.

Naturalism explains some cognitive adaptation but not the prior mathematical organization of the universe.

Computationalism models transformation while presupposing logic, state distinction, hardware regularity, and executable law.

The resulting landscape is plural but not unified. Each school begins with a different unexplained remainder.

1.4 The Mathematical Intelligibility of Physics

The success of mathematics in physics presents a stronger problem than practical usefulness.

Physical theories repeatedly employ mathematical structures developed without immediate empirical motivation. Symmetry groups, differential geometry, Hilbert spaces, topology, and category-theoretic structures later become indispensable descriptions of nature.

This correspondence suggests more than convenient notation. It indicates a shared intelligible order between:

the structure of reality,

the structure of mathematics,

and the structure of rational cognition.

A worldview that excludes any common ground must treat this convergence as accidental, emergent, or brute.

1.5 Restoring the Logos

The proposed ground is the Divine Logos.

The Logos is not one object within mathematics. It is the source of the conditions under which mathematical objecthood is possible:

being,

identity,

distinction,

relation,

law,

coherence,

and intelligibility.

The foundational dependence may be written:

Logos $\vdash$ mathematics.

The turnstile denotes ontological grounding, not formal derivation. Mathematics follows from an intelligible order sustained by the Logos; it is not produced by manipulating an empty symbol system.

The Divine spark is the corresponding capacity of finite mind to participate in this order. Mathematical discovery occurs because intelligibility in the knower is responsive to intelligibility in what is known.

The foundational sequence is therefore:

Logos → intelligible possibility → lawful relation → mathematical objecthood → representation → proof.

The formal theorem developed below begins at lawful relation. The Logos-centered framework explains why that starting point exists.

2. The Divine Logos as the Ground of Mathematics

2.1 Logos, Being, and Intelligibility

Mathematics presupposes that something can be, remain itself, differ from another, and stand in lawful relation.

These are not late products of notation. They are prior conditions of notation.

The Logos grounds this prior intelligibility. Mathematical truth is possible because being is not chaotic, identity is not arbitrary, and relation is not accidental.

Thus:

Logos → intelligible order.

Mathematics is the formal articulation of that order.

2.2 Identity and Distinction Before Formalization

Every formal system requires stable identity.

A symbol must remain the same symbol across an inference. An object must be distinguishable from another object. Equality must differ from inequality. A rule must preserve its meaning through repeated use.

Therefore identity and distinction precede all calculation.

They cannot be produced by a system that already depends on them.

The Logos-grounded account treats identity and distinction as ontological gifts rather than unexplained formal conveniences.

2.3 Relation and Law as Ontological Categories

A relation is not merely a mark placed between two terms.

It requires:

1. distinguishable relata;
2. a stable mode of connection;
3. lawful persistence;
4. intelligibility to a knower.

Mathematics abstracts these relations, but does not create their possibility.

The Divine Spark framework therefore places relation between pure possibility and determinate objecthood:

intelligible possibility → lawful relation → actual object.

Relation is the principle by which a possible structure becomes a definite mathematical world.

2.4 Mathematical Truth as Participation

Mathematical knowledge is neither arbitrary invention nor passive copying.

The mind constructs definitions, symbols, and proofs. Yet those constructions succeed only when they participate in a prior intelligible order.

A theorem is discovered because the consequences are not freely chosen after the definitions are fixed.

The human mind contributes formulation.

The Logos contributes truth.

Mathematical discovery is therefore participation in intelligibility.

2.5 The Objective and Subjective Divine Spark

The Divine spark has two aspects.

The objective aspect is the intelligible order present in reality.

The subjective aspect is the rational capacity of finite minds to recognize and extend that order.

These are not independent miracles. They correspond because both are grounded in the Logos.

Hence:

Logos in reality ↔ Logos-responsive mind.

This correspondence explains why abstract mathematics can disclose physical law and why physical regularity can be expressed mathematically.

The Divine spark is not a hidden variable inside an equation. It is the condition under which equations can be true, understood, and discovered.

3. From Logos to Mathematical Object-Worlds

3.1 The Intelligible Possibility Space Ω

Let Ω denote a lawful space of mathematical possibilities.

Ω is not an unstructured void. It already presupposes identity, distinction, admissible relation, and coherent transformation.

Under the Logos-centered interpretation:

Logos ⊢ Ω.

The symbol Ω therefore names intelligible possibility, not mathematical nothingness.

3.2 Actualizing Relation Policies π

A relation policy π specifies which distinctions remain significant and which are identified.

Formally, π may be an equivalence relation, congruence, closure rule, presentation, or family of admissibility constraints.

Its role is selective:

π : Ω → lawful identifications.

A policy does not create intelligibility. It organizes intelligible possibilities into a determinate object-world.

3.3 Quotient Objecthood

The resulting object is represented as

$A = \Omega/\pi$.

The quotient does not merely discard information. It establishes objecthood by declaring which differences are irrelevant under π.

Thus identity inside A is policy-indexed:

$x \equiv_\pi y$.

Mathematical objects are therefore not bare collections of elements. They are stabilized patterns of distinction and identification.

3.4 The Logos-Grounded Judgment

The complete judgment is

$\text{Logos} \vdash A = \Omega/\pi$.

This expression contains two irreducible levels.

The turnstile records ontological dependence: the possibility of Ω, π, equality, and quotient formation is grounded in the Logos.

The equation records formal construction: within that intelligible order, A is obtained by quotienting Ω under π.

Confusing these levels produces two opposite errors.

The first treats God as a variable inside the formal system.

The second treats the formal system as self-grounding.

3.5 Discovery, Construction, and Participation

The quotient model unifies three accounts of mathematics.

Construction occurs because a mathematician chooses π.

Discovery occurs because the consequences of π are not freely chosen.

Participation occurs because both Ω and the rational capacity to investigate it belong to the intelligible order of the Logos.

Thus mathematical activity has the form:

receive possibility → impose lawful distinction → discover necessary consequence.

The finite semilattice theorem developed below quantifies how the complexity of an actualized object-world depends on the number of generators and actualizing relations.

4. Why Secular Foundations Remain Incomplete

4.1 Platonism Without an Ultimate Ground

Platonism preserves the objectivity of mathematics by treating mathematical truths as independent of human invention.

Its difficulty is not objectivity, but grounding.

If mathematical forms exist timelessly, one must still ask:

Why do they exist?

Why are they internally coherent?

Why can physical reality instantiate them?

Why can finite minds know them?

A Platonic realm names the location of mathematical truth without explaining its source.

The Logos-centered account retains mathematical objectivity while grounding it:

Logos $\vdash$ mathematical form.

4.2 Formalism Without Truth

Formalism identifies mathematics with symbol systems governed by explicit rules.

This secures rigor but cannot fully explain truth.

A formal derivation shows that a sentence follows from selected axioms. It does not explain:

why those axioms are intelligible;

why inference preserves truth;

why one system is mathematically fruitful and another sterile;

or why formal structures describe physical reality.

Syntax cannot generate its own semantics.

Formal systems are instruments of mathematical reason, not the ontological source of mathematical truth.

4.3 Structuralism Without the Source of Structure

Structuralism correctly recognizes that mathematical objects are often determined by their relations rather than by intrinsic substance.

The natural numbers, for example, are characterized by their structural position and lawful relations.

Yet structure still presupposes:

distinguishable positions;

stable relations;

coherent transformation;

and identity under isomorphism.

To say that structure is fundamental does not explain why lawful structure exists.

The Divine Spark framework accepts the priority of relation while grounding relational intelligibility in the Logos.

4.4 Constructivism Without an Ultimate Constructor

Constructivism requires mathematical objects to arise through admissible constructions.

This exposes an important truth: mathematical existence is connected to lawful formation.

But construction already presupposes:

a constructor;

permitted operations;

stable procedures;

and standards of valid completion.

The ultimate origin of constructibility therefore remains unexplained.

In the Logos-centered account, human construction is secondary participation in a prior intelligible order. The mathematician constructs within possibility; the mathematician does not create the possibility of law itself.

4.5 Naturalism and the Prior Intelligibility of Nature

Naturalism may explain mathematical cognition as an evolved capacity for recognizing patterns relevant to survival.

This may account for elementary estimation, spatial judgment, and regularity detection.

It does not adequately explain:

the prior mathematical structure of the world;

the applicability of highly abstract mathematics;

the apparent necessity of mathematical truths;

or the correspondence between independently developed mathematics and fundamental physics.

Evolution may explain why a mind notices order.

It does not explain why there is order to notice.

4.6 Computationalism and Hidden Mathematical Inheritance

Computationalism proposes that mathematical objects or theories may emerge through executable processes.

But every computation inherits a mathematical substrate:

distinguishable states;

logical gates;

transition rules;

temporal order;

identity of memory locations;

and lawful hardware behavior.

A program cannot generate mathematics from nonmathematics because its execution already depends on mathematical structure.

Computation can actualize, search, compress, and reveal mathematical worlds.

It cannot explain the ultimate origin of intelligibility.

The correct hierarchy is therefore:

Logos → intelligible order → formal system → computation → discovered consequence.

Computation belongs within mathematics. It is not the ground beneath mathematics.

5. Finite Meet-Semilattices as a Formal Model

5.1 Free Meet-Semilattices and Generator Supports

Let G = {1, …, g} index generators x_1, …, x_g.

The free finite meet-semilattice on G may be identified with the nonempty subsets of G. A subset U ⊆ G represents the term

x_U = ⋀{x_i : i ∈ U}.

Because meet is associative, commutative, and idempotent, every term is determined uniquely by its support U.

Meet becomes union:

x_U ∧ x_V = x_(U ∪ V).

Thus formal syntax collapses to finite support geometry.

5.2 Defining Equations as Hitting Conditions

A presentation consists of equations

U_k = V_k,

for 1 ≤ k ≤ s.

Let 2 = {0,1} be the two-element meet-semilattice.

A homomorphism into 2 is determined by its zero set

Z = {i ∈ G : x_i ↦ 0}.

For a support U,

x_U ↦ 0

exactly when

U ∩ Z ≠ ∅.

Therefore Z satisfies U = V exactly when

U ∩ Z ≠ ∅ ⇔ V ∩ Z ≠ ∅.

An admissible zero set must hit both sides or neither side.

Each equation is therefore a lawful constraint on possible binary viewpoints.

5.3 Zero Sets and Two-Element Homomorphisms

Let X_R denote the family of zero sets satisfying every equation in a presentation R.

Then

Hom(A,2) ≅ X_R.

Every admissible zero set determines one binary coordinate on A, and every binary coordinate arises in this way.

The formal object A is therefore represented by the family of admissible distinctions it permits.

This gives a finite realization of the passage

possibility → relation → representation.

The generators supply possible distinctions.

The equations determine which distinctions may coexist.

The homomorphisms record the resulting binary viewpoints.

5.4 Union-Closed Solution Families

The family X_R is closed under union.

If Z_1 and Z_2 satisfy every defining equation, then $Z_1 \cup Z_2$ also satisfies every equation.

Indeed, a support misses $Z_1 \cup Z_2$ exactly when it misses both Z_1 and Z_2.

Thus X_R forms a finite join-semilattice under union.

The indispensable binary coordinates are the proper join-irreducible members of X_R.

A coordinate is join-irreducible when it cannot be reconstructed as the union of two strictly smaller admissible coordinates.

Hence irreducibility expresses representational necessity.

5.5 Mathematical Actualization in the Finite Model

The free meet-semilattice contains the initial combinatorial possibilities.

The equations impose a relation policy.

The quotient stabilizes one mathematical object-world.

The admissible zero sets describe its lawful binary representations.

In schematic form:

$F_g \to R \to A \to \mathrm{Hom}(A,2)$.

Within the broader framework:

$\mathrm{Logos} \vdash F_g/R$.

The Logos does not replace the presentation or proof. It grounds the prior intelligibility of generators, equality, relation, and lawful consequence.

The finite model therefore isolates one exact form of mathematical actualization:

A relation budget transforms a space of possible distinctions into a determinate object whose representational complexity can be counted.

6. Generator Rank, Relation Rank, and Subdirect Dimension

6.1 Minimum Generator Rank γ(A)

For a finite meet-semilattice A, define

$\gamma(A) = \min\{|S| : S \text{ generates } A\}$.

This is the least number of primitive elements required to recover every element of A by finite meets.

Generator rank measures available mathematical possibility before additional relations are imposed.

A presentation may contain redundant generators. The invariant γ(A) removes this redundancy.

6.2 Minimum Relation Rank $\sigma_2(A)$

Fix a generating set of minimum size γ(A).

Define $\sigma_2(A)$ as the least number of equations required to present A on such a generating set.

Thus

$\sigma_2(A) = \min\{|R| : A \cong F_g/\theta_R \text{ and } g = \gamma(A)\}$.

Relation rank measures the least relational input needed to stabilize the object-world A.

Redundant equations do not contribute.

6.3 Two-Element Subdirect Dimension $\nu_2(A)$

Let 2 = {0,1} be the two-element meet-semilattice.

Define $\nu_2(A)$ as the least integer N for which A admits a subdirect embedding

$A \hookrightarrow 2^N$.

Equivalently, $\nu_2(A)$ is the minimum number of surjective homomorphisms $A \to 2$ required to separate all elements of A.

This invariant measures representational complexity.

It counts the irreducible binary viewpoints required to recover the object faithfully.

6.4 The Extremal Function $M_2(g,s)$

Define

$M_2(g,s) = \max\{\nu_2(A) : \gamma(A) = g \text{ and } \sigma_2(A) \leq s\}$.

This asks:

How large can representational complexity become when the object-world has g essential generators and at most s essential relations?

The answer is

$M_2(g,s) = \Theta(g^{(s+1)})$

for fixed s.

Relation rank therefore controls the degree of extremal growth.

6.5 Possibility, Constraint, and Representation

The three invariants form a natural hierarchy:

$\gamma(A)$ measures possibility.

$\sigma_2(A)$ measures actualizing constraint.

$\nu_2(A)$ measures necessary representation.

Thus:

generator rank → relation rank → subdirect dimension.

In the Logos-centered interpretation:

Logos grounds possibility.

Relations actualize objecthood.

Binary coordinates disclose the resulting object.

The formal problem is therefore not merely one of compression. It measures how much representational structure can be actualized by a bounded relational act.

7. The Root-and-Repair Principle

7.1 Join-Irreducible Coordinates

Let X_R be the union-closed family of admissible zero sets determined by a presentation R.

A proper solution Z in X_R is join-irreducible when

$Z = Y \cup W$

implies

$Z = Y$ or $Z = W$.

Such a coordinate cannot be reconstructed from strictly smaller admissible coordinates.

The minimum two-element subdirect dimension is therefore the number of proper join-irreducible solutions:

$v_2(A) = |J^*(X_R)|$.

The problem reduces to bounding the size and number of these indispensable supports.

7.2 The Irreducible Root

For a proper join-irreducible solution Z, let Z^- be the union of all proper admissible subsets of Z.

Because X_R is union-closed, Z^- is admissible.

Join-irreducibility implies

$Z^- \subsetneq Z$.

Choose

$r \in Z \setminus Z^-$.

Then no proper admissible subset of Z contains r.

The element r is a root of Z.

Once an admissible reconstruction contains r and remains inside Z, it must equal Z.

7.3 Violated Relations

Begin with the singleton

$Y_0 = \{r\}$.

The set Y_0 need not satisfy the presentation.

Suppose the current set Y violates an equation U = V. Then Y hits exactly one side.

Assume

$Y \cap U \neq \emptyset$

and

$Y \cap V = \emptyset$.

Because $Y \subseteq Z$ and Z satisfies U = V, the set Z must hit V.

Thus there exists

$v \in Z \cap V$.

Adding v repairs the violated equation.

7.4 One Permanent Repair per Relation

Replace Y by

$Y \cup \{v\}$.

The repaired set now hits both U and V.

Every later enlargement also hits both sides. Therefore the same equation can never again become violated.

Each defining equation requires at most one repair.

If the presentation contains s equations, the reconstruction uses at most s added elements.

7.5 The Support Bound

Starting from one root and performing at most one repair per relation gives an admissible set Y* satisfying

$r \in Y^* \subseteq Z$

and

$|Y^*| \leq s + 1$.

Because no proper admissible subset of Z contains r,

$Y^* = Z$.

Therefore every indispensable coordinate satisfies

$|Z| \leq s + 1$.

This is the root-and-repair lemma.

7.6 Relation as a Controlled Actualizing Principle

The bound has a direct structural interpretation.

One root initiates an irreducible coordinate.

Each relation may force one additional distinction into that coordinate.

No relation contributes more than one permanent repair.

Thus a budget of s relations can actualize indispensable supports of size at most s + 1.

The result is local, but its consequence is global: the relation budget determines the maximal polynomial degree of representational complexity.

8. The Divine Spark Complexity Law

8.1 The Finite Binomial Upper Bound

Every indispensable zero set has size at most s + 1.

Therefore

$\nu_2(A) \leq \sum$ from j = 1 to s + 1 of C(g,j).

For fixed s,

$\sum$ from j = 1 to s + 1 of C(g,j)

has leading order

g^(s+1)/(s+1)!.

Hence

$M_2(g,s) = O(g^{(s+1)})$.

The upper bound is purely combinatorial. It follows from the number of supports that can be built from one root and at most s repairs.

8.2 Balanced Block-Chain Constructions

Partition the g generators into s + 1 nonempty blocks

$V_0, V_1, \ldots, V_s,$

with block sizes

$b_0, b_1, \ldots, b_s.$

Let q_i be the meet of all generators in V_i.

Impose the chain of equations

$q_0 = q_1,$

$q_1 = q_2,$

…

$q_(s-1) = q_s.$

A nonempty admissible zero set must hit every block.

The proper join-irreducible solutions are exactly the transversals containing one generator from each block.

Therefore

$\nu_2(A) = b_0 b_1 \cdots b_s.$

When the blocks are balanced,

$b_i \approx g/(s+1),$

and therefore

$\nu_2(A) \approx [g/(s+1)]^{(s+1)}.$

Thus

$M_2(g,s) = \Omega(g^{(s+1)}).$

8.3 The Sharp Extremal Degree

The upper and lower bounds have the same degree:

$M_2(g,s) = O(g^{(s+1)})$

and

$M_2(g,s) = \Omega(g^{(s+1)})$.

Therefore

$M_2(g,s) = \Theta(g^{(s+1)})$

for every fixed nonnegative integer s.

Equivalently,

lim as $g \to \infty$ of

$\log M_2(g,s) / \log g$

equals

$s + 1$.

The exponent is exact even when the leading constant is not yet known.

8.4 The Spark-Degree Theorem

Theorem

For finite meet-semilattices of minimum generator rank g and relation rank at most s,

$M_2(g,s) = \Theta(g^{(s+1)})$

for fixed s.

Proof

The root-and-repair lemma bounds every indispensable support by s + 1 generators. Counting such supports gives the polynomial upper bound.

Balanced block-chain constructions produce on the order of $g^{(s+1)}$ indispensable transversals while using s defining relations.

The degrees coincide. ■

8.5 One Additional Relation and One Additional Degree

The extremal hierarchy is

$M_2(g,0) = \Theta(g)$,

$M_2(g,1) = \Theta(g^2)$,

$M_2(g,2) = \Theta(g^3)$,

and generally

$M_2(g,s) = \Theta(g^{(s+1)})$.

Thus each additional allowed relation raises the extremal polynomial degree ceiling by one.

This statement is not monotonic for individual algebras. A particular new relation may collapse distinctions or be redundant.

The law concerns maximal possible complexity under a relation budget.

8.6 Formal Theorem and Metaphysical Interpretation

The formal theorem states:

relation rank controls the degree of extremal two-state subdirect complexity.

The metaphysical interpretation states:

lawful relation is an actualizing principle within a Logos-grounded field of mathematical possibility.

These claims are distinct but compatible.

The proof requires no theological axiom.

The theorem alone, however, does not explain why generators, equality, relation, law, or intelligibility exist.

The complete statement is therefore two-level:

Logos $\vdash$ lawful mathematical possibility,

and within that possibility,

$M_2(g,s) = \Theta(g^{(s+1)})$.

The Divine Spark Complexity Law names this union of ontological grounding and exact formal consequence.

9. The Exact One-Relation Theorem

9.1 Four Regions of a Single Relation

Consider one defining equation

$U = V$.

Partition the generator set G into four regions:

$A = U \setminus V$,

$B = V \setminus U$,

$C = U \cap V$,

$R = G \setminus (U \cup V)$.

Let

$a = |A|$,

$b = |B|$,

$c = |C|$,

$r = |R|$,

so that

$a + b + c + r = g$.

The regions have distinct logical roles.

A hits only the left side.

B hits only the right side.

C hits both sides simultaneously.

R hits neither side.

This four-part decomposition completely determines the admissible zero-set geometry of a one-relation presentation.

9.2 Classification of Indispensable Coordinates

The proper join-irreducible zero sets are exactly the following:

1. Singletons from R.
2. Pairs containing one element of A and one of B.
3. Singletons from C.
4. Pairs containing one element of C and one of A.
5. Pairs containing one element of C and one of B.

These classes contribute, respectively,

r,

ab,

c,

ca,

cb

coordinates.

No other admissible zero set is irreducible.

A set containing several outside generators decomposes into outside singletons.

A set containing excess elements from A or B decomposes into smaller admissible hitting sets.

A set containing a common element together with elements from both asymmetric sides decomposes through the common singleton.

Thus the classification is complete.

9.3 Exact Coordinate Count

Summing the five classes gives

$v_2 = r + ab + c + ca + cb.$

Therefore

$v_2 = r + ab + c(a + b + 1).$

Using

$r = g - a - b - c,$

one may also write

$v_2 = g - a - b + ab + c(a + b).$

This formula is exact for every nontrivial one-relation presentation.

The entire subdirect complexity is encoded by four region sizes.

9.4 Elimination of Outside Generators

At an extremum, outside generators are inefficient.

Moving one generator from R into C changes the dimension by

$a + b$.

For a nontrivial equation, $a + b > 0$.

Therefore the dimension strictly increases.

Hence every extremal one-relation presentation satisfies

$r = 0$.

The optimization reduces to

$a + b + c = g$

and

$v_2 = ab + c(a + b + 1)$.

The extremal relation uses every generator.

9.5 The Balanced-Square Identity

Under the constraint

$a + b + c = g$,

the exact coordinate count satisfies

$2v_2 = g^2 + 1 - [a^2 + b^2 + (c - 1)^2]$.

Thus maximizing v_2 is equivalent to minimizing

$a^2 + b^2 + (c - 1)^2$.

The three integers

a,

b,

$c - 1$

have fixed sum

$g - 1$.

Their sum of squares is minimized when they differ by at most one.

The exact extremal problem therefore reduces to the elementary principle of balanced integers.

9.6 Exact Formula

Write

$g - 1 = 3q + t,$

where $t \in \{0,1,2\}$.

The minimizing triples are the balanced permutations of:

$t = 0$:

(q, q, q),

$t = 1$:

$(q + 1, q, q)$,

$t = 2$:

$(q + 1, q + 1, q)$.

Substitution yields the unified formula

$M_2(g,1) = \text{floor}((g^2 + g + 1)/3)$,

for $g \geq 3$.

Thus

$M_2(g,1) = g^2/3 + g/3 + O(1)$.

The exact quadratic coefficient is 1/3.

This exceeds the balanced two-block coefficient 1/4, showing that the simplest block-chain construction proves the correct degree but not the optimal constant.

The one-relation theorem therefore supplies both an exact extremum and a warning: extremal mathematical actualization may require overlapping relational geometry rather than disjoint block structure.

10. The Polynomial-to-Exponential Threshold

10.1 Binomial-Sum Control

For every finite meet-semilattice A,

$\nu_2(A) \leq \Sigma$ from $j = 1$ to $\sigma_2(A) + 1$ of $C(\gamma(A), j)$.

Set

$g = \gamma(A)$

and

$s = \sigma_2(A)$.

Then

$\nu_2(A) \leq \Sigma$ from $j = 0$ to $s + 1$ of $C(g, j)$.

This is an initial segment of the binomial expansion of 2^g.

Its asymptotic scale depends on the relation density

$\alpha = (s + 1)/g$.

Fixed s gives polynomial growth.

Sublinear s gives subexponential growth.

Linear s permits exponential growth.

Thus relation rank controls not only polynomial degree, but the transition between complexity regimes.

10.2 Entropy Bound

For $0 \leq \alpha \leq 1/2$,

Σ from $j = 0$ to αg of $C(g, j) \leq 2^{(gH(\alpha))}$,

where

$H(\alpha) = -\alpha \log_2 \alpha - (1 - \alpha) \log_2(1 - \alpha)$

is the binary entropy function.

Therefore, when

$s + 1 \leq g/2$,

$\nu_2(A) \leq 2^{[gH((s + 1)/g)]}$.

For $\alpha < 1/2$,

$H(\alpha) < 1$.

Hence the representation dimension remains strictly below the full Boolean scale 2^g.

10.3 Sublinear Relation Rank

Suppose

$s = o(g)$.

Then

$(s + 1)/g \to 0$.

Since $H(\alpha) \to 0$ as $\alpha \to 0$,

$\log_2 v_2(A) = o(g)$.

Therefore

$v_2(A) = 2^{o(g)}$.

Sublinear relation rank forces subexponential two-element subdirect dimension.

Several regimes follow immediately.

If s is constant, $v_2(A)$ is polynomial.

If $s = O(\log g)$, then

$\log v_2(A) = O((\log g)^2)$.

If $s = O(g^\beta)$ for $0 < \beta < 1$, then

$\log v_2(A) = O(g^\beta \log g)$.

None reaches exponential scale.

10.4 Necessity of Linear Relation Density

Suppose a family A_g satisfies

$v_2(A_g) \geq 2^{cg}$

for some constant $c > 0$.

Then the entropy bound implies that

$\sigma_2(A_g)/\gamma(A_g)$

cannot tend to zero.

Therefore

$\sigma_2(A_g) = \Omega(\gamma(A_g))$.

Exponential representational complexity requires relation rank linear in generator rank.

This is a necessary condition.

It means that exponentially many indispensable binary viewpoints cannot arise from a vanishingly sparse relational budget.

10.5 Sufficiency in Scale

Linear relation density is also sufficient for exponential growth in order of magnitude.

Take s + 1 blocks of approximately equal bounded size.

If

$s + 1 \approx \alpha g$,

then each block has size approximately $1/\alpha$.

The block-chain construction gives

$\nu_2(A) \approx (1/\alpha)^{(\alpha g)}$.

Hence

$\nu_2(A) = 2^{[\alpha g \log_2(1/\alpha)]}$.

For fixed $\alpha > 0$, this is exponential in g.

Thus the threshold is sharp in scale:

$s = o(g)$ implies subexponential complexity,

while

$s = \Theta(g)$ permits exponential complexity.

10.6 Creation, Constraint, and Complexity

The threshold admits a direct interpretation.

Generators provide potential distinctions.

Relations bind those distinctions into irreducible combinations.

Sparse relation budgets actualize only low-support coordinates.

Dense relation budgets permit exponentially many high-support coordinates.

The formal transition is:

fixed relation rank → polynomial complexity,

sublinear relation rank → subexponential complexity,

linear relation rank → exponential complexity.

Within the Logos-centered framework, complexity does not emerge from lawlessness.

It emerges from increasingly dense lawful relation.

The transition to exponential scale is therefore not a transition from order to disorder. It is a transition from sparse order to relationally dense order.

11. Mathematics, Physics, and the Recovery of Unity

11.1 Why Mathematics Describes Physical Reality

The applicability of mathematics to physics is not explained by formalism alone.

A formal system can generate consequences from axioms. It cannot explain why physical reality should instantiate those consequences.

The correspondence becomes intelligible if mathematics and physics arise within one Logos-grounded order.

Then mathematical law and physical law are not strangers accidentally matched after the fact. They are distinct disclosures of one intelligible reality.

Thus:

Logos → mathematical intelligibility,

and

Logos → physical lawfulness.

Their correspondence follows from common origin.

11.2 Lawfulness Across Scales

Physics exhibits lawful structure across scales:

quantum,

molecular,

biological,

planetary,

and cosmological.

The governing descriptions differ, but the expectation of coherence remains.

Scientists search for invariants, symmetries, conservation laws, equivalences, and transformations because reality is assumed to remain intelligible under change.

This assumption is rarely proved. It is foundational.

A Logos-centered physics makes the assumption explicit:

reality is expected to be coherent because it participates in one rational source.

11.3 Observer Intelligibility and Participation in Logos

A physical theory requires more than lawful objects.

It also requires observers capable of recognizing law.

The correspondence is therefore threefold:

mathematical structure,

physical structure,

rational cognition.

A purely external account of physics leaves unexplained why minds can form concepts adequate to nature.

The Divine spark names the finite mind's capacity to participate in the same intelligibility expressed objectively in physical law.

Thus observation is not the collision of an unintelligible world with an accidental calculator.

It is encounter within a shared order.

11.4 Relation-First Approaches to Physical Unification

Modern physics repeatedly shifts from isolated substances toward relations:

fields rather than particles alone,

symmetries rather than lists of forces,

gauge equivalence rather than unique descriptions,

reference frames rather than absolute coordinates,

entanglement rather than separable states.

This movement is compatible with the present framework.

If objecthood is stabilized by lawful relation, then physical entities may be secondary to the invariant relational structures through which they become identifiable.

The formal semilattice model does not establish a physical theory.

It provides a finite example of a general pattern:

possibility → relation → objecthood → representation.

11.5 Limits of God-Excluded Foundations

A foundation that excludes God may still produce accurate local theories.

Its limitation appears at the level of ultimate explanation.

It cannot fully account for:

why laws exist;

why they are coherent;

why mathematics describes them;

why rational minds understand them;

why truth should be unified;

or why deeper unification should be expected.

These questions are often declared metaphysical and set aside.

But setting aside a question is not answering it.

The resulting physics may remain predictively successful while ontologically fragmented.

11.6 Toward a Logos-Grounded Mathematical Physics

A Logos-grounded mathematical physics would not insert theological terms into empirical equations.

It would order the levels correctly.

At the ontological level:

Logos grounds being, identity, relation, and law.

At the mathematical level:

formal structures articulate lawful possibility.

At the physical level:

selected structures describe instantiated reality.

At the epistemic level:

finite minds participate in and discover that order.

The program may be summarized:

Logos ⊢ mathematical possibility,

Logos ⊢ physical law,

therefore mathematical participation in physical reality is intelligible.

This does not eliminate technical disagreement.

It restores the expectation that mathematics and physics belong to one coherent reality rather than to disconnected domains joined by unexplained coincidence.

12. Discussion and Conclusion

12.1 What the Formal Theorem Establishes

The formal result is exact in its stated domain.

For finite meet-semilattices,

$\nu_2(A) \leq \Sigma$ from j = 1 to $\sigma_2(A) + 1$ of C(γ(A), j).

For fixed relation rank s,

$M_2(g,s) = \Theta(g^{(s+1)})$.

For one relation,

$M_2(g,1) = \text{floor}((g^2 + g + 1)/3)$.

The polynomial-to-exponential threshold follows from the same support bound:

sublinear relation rank implies subexponential two-element subdirect dimension, while exponential dimension requires linear relation density.

These are finite algebraic theorems. Their validity does not depend on theological assent.

12.2 What the Theorem Does Not Prove

The theorem does not prove the existence of God.

It does not derive the Logos from semilattice combinatorics.

It does not show that every mathematical relation is divinely selected.

It does not establish that finite meet-semilattices are a complete model of mathematical creation.

It does not convert theology into an empirical variable.

The theological interpretation and the formal theorem must remain logically distinct.

The theorem explains how bounded relations control actualized representational complexity within a lawful mathematical domain.

It does not explain why lawful mathematical domains exist.

12.3 Why the Logos Interpretation Is Still Necessary

A complete account of mathematics must address both internal structure and ultimate ground.

The theorem answers an internal question:

Given generators, equations, equality, and valid inference, how does relation rank control subdirect complexity?

The Logos-centered account addresses the prior question:

Why are identity, distinction, relation, law, truth, and intelligibility available at all?

Without the second level, the first begins from unexplained primitives.

The formal system remains useful, but its ontology remains incomplete.

Thus the two levels are complementary:

formal proof explains consequence;

Logos explains intelligible possibility.

12.4 Extensions Beyond Semilattices

The semilattice case is unusually tractable because terms reduce to supports, equations become hitting conditions, and admissible zero sets form a union-closed family.

The broader research problem is to identify other algebraic systems in which indispensable coordinates admit bounded repair complexity.

One may seek a general inequality of the form

support complexity ≤ initial seed size + repair number.

If the repair number grows linearly with relation rank, a corresponding polynomial degree law may follow.

Possible extensions include:

finite lattices,

Horn algebras,

closure systems,

idempotent commutative structures,

fixed-core representations,

and relation-governed categorical quotients.

The exponent need not remain s + 1. The semilattice theorem provides the first exact model.

12.5 Open Mathematical Problems

The principal finite-algebra questions are:

Determine $M_2(g,s)$ exactly for $s \geq 2$.

Determine the optimal leading constant in

$M_2(g,s) \sim c_s g^{(s+1)}$.

Classify extremal presentations.

Determine whether overlapping relations outperform balanced block chains for every $s \geq 1$.

Find the exact exponential rate when s is proportional to g.

Extend the theory to finite cores larger than 2.

Characterize signatures admitting a root-and-repair principle.

These are mathematical problems independent of the theological interpretation.

12.6 From the Logos, Through Relation, Mathematical Worlds Become Actual

The completed hierarchy is:

Logos → intelligible possibility → lawful relation → mathematical objecthood → representation → proof.

The finite theorem occupies the middle of this sequence.

Generators provide possible distinctions.

Relations bind them into stable object-worlds.

Irreducible coordinates disclose the object from elementary viewpoints.

The number of such viewpoints obeys the Spark-Degree Law:

$M_2(g,s) = \Theta(g^{(s+1)})$.

The broader Divine Spark Complexity Law states that this formal actualization does not occur in an ontological vacuum.

Mathematical possibility is intelligible because it is grounded in the Logos.

Mathematical discovery is possible because finite minds participate in that intelligibility.

Mathematical worlds become actual through lawful relation.

The final statement is therefore:

Logos $\vdash A = \Omega/\pi$,

and within the finite semilattice model,

$M_2(g,s) = \Theta(g^{(s+1)})$.

From the Logos, through relation, mathematical worlds become actual.

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